Sec 13.3

D: the Laplace operator associated with the form $(.|.)$

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \qquad (v_1 \dots v_l, \Lambda_0, \delta) \to y$$

we have in coordinates (13.2.4) $\left(v = 2\pi i\left(\sum_{s=1}^{l} z_s v_s - \tau\Lambda_0 + u\delta\right)\right)$

(13.3.1) $$D = \frac{1}{4\pi^2}\left(2\frac{\partial}{\partial u}\frac{\partial}{\partial \tau} - \sum_{s=1}^{l}\left(\frac{\partial}{\partial z_s}\right)^2\right)$$

Since $D(e^{\lambda}) = (\lambda|\lambda)e^{\lambda}$

We deduce from (13.2.1) $\left(\Theta_\lambda = e^{-\frac{(\lambda|\lambda)}{2k}\delta} \sum_{\alpha\in M} e^{t_\alpha(\lambda)}\right)$

$\Theta_\lambda = \Theta_{\lambda + a\delta}$

$$D(\Theta_\lambda) = D(\Theta_{\underbrace{\lambda - \frac{(\lambda|\lambda)}{2k}\delta}_{\lambda'}}) = D\left(\sum_{\alpha\in M} e^{t_\alpha(\lambda')}\right)$$

$(\lambda'|\lambda') = 0$

$$= \sum_{\alpha\in M} (\lambda'|\lambda')\, e^{t_\alpha(\lambda')} = 0$$

(13.3.2).

Recall. $F \longrightarrow K$ $\begin{cases} T_1: & F(n.v) = F(v) \text{ all } n \in N_{\mathbb{Z}} \\ T_2: & F(v + a\delta) = e^{ka}F(v) \text{ for all } a \end{cases}$

$\widehat{Th}_k \qquad \widehat{Th} = \bigoplus_{k\geq 0} \widehat{Th}_k$

$D(\Theta_\lambda) = 0$

We put $Th_0 = \mathbb{C}$. $Th_k = \{F \in \widehat{Th}_k \mid D(F) = 0\}$ for $k > 0$

Rmk: $Th = \bigoplus_{k\geq 0} Th_k$ is not a subring of $\widehat{Th}$

prop 13.2 : $\theta_\lambda \cdot \theta_\mu = \sum_{\alpha \in M \ldots} \underbrace{\theta_{\lambda+\mu+n\alpha}}_{D(\ \ ) \neq 0} \psi_\alpha$

$\theta_\lambda \to m$, $\theta_\mu \to n$

prop 13.3. The set $\{ \theta_\lambda , \lambda \in P_k \bmod (kM + \mathbb{C}\delta) \}$ is a $\mathbb{C}$-basis of Th_k ($\widetilde{Th}_0$ - basis of $\widetilde{Th}_k$) for $k > 0$.

Pf: Let $F \in \widetilde{Th}_k$ using $F(P_\alpha(v)) = F(v)$ for all $\alpha \in M$ (T_1)

$N = \mathfrak{h}^0_{\mathbb{R}} \times \mathfrak{h}^0_{\mathbb{R}} \times i\mathbb{R}$ $\quad P_\alpha = (\alpha, 0, 0) \in N$. $\quad v + 2\pi i \alpha$ $\quad v \in Y$ (τ, z, u)

We can, for a fixed τ, decompose F into a Fourier series.

+ (by T_2) $\quad F = e^{k\Lambda_0} \sum_{\gamma \in M^*} a_\gamma(\tau) e^{\gamma}$ $\quad (13.3.3)$

as e^{ka} $\quad a_\gamma$

By (T_1) $a_\gamma e^{\frac{(\gamma|\gamma)}{2k}\delta}$ depends only on $\gamma \bmod kM$. It follow that

$$\boxed{F = \sum_{\lambda \in P_k \bmod kM + \mathbb{C}\delta} c_\gamma \theta_\lambda}$$

$(0, \alpha, 0)(u) \in N_2$

$F(t_\alpha(u)) = F(u) \qquad t_\alpha(u) = u + (u|\delta)\alpha$

$$((e^{-k\Lambda_0} F)(t_\alpha(u)) = e^{(-k\Lambda_0 | t_\alpha(u))} F(t_\alpha(u))$$

$$= e^{-k(\Lambda_0|u) + k(u|\alpha) + k(\alpha|\alpha)(u|\delta)/2} F(u)$$

$$= e^{k(u|\alpha) + k(\alpha|\alpha)(u|\delta)/2} (e^{-k\Lambda_0} F)(u)$$

i.e $\sum_{\gamma \in M^*} a_\gamma e^{\gamma + (\gamma|\alpha)\delta}(u) = \sum_{\gamma \in M^*} a_\gamma e^{k\alpha + k(\alpha|\alpha)\delta/2 + \gamma}$ $\quad \gamma = \gamma - k\alpha$

$$\sum a_\gamma e^{\gamma + (\gamma|\lambda)\delta} = \sum_{\gamma} a_{\gamma - k\alpha} e^{\frac{k(\alpha|\alpha)}{2} \cdots}$$

$$\Rightarrow \boxed{a_\gamma = a_{\gamma - k\alpha}\, e^{\frac{k(\alpha|\alpha)\delta}{2} - (\gamma|\alpha)\delta}}$$

Hence $a_\gamma e^{\frac{(\gamma|\gamma)\delta}{2k}} = a_{\gamma - k\alpha}\, e^{(\gamma - k\alpha|\gamma - k\alpha)\delta/2k}$

let $c_\gamma = a_\gamma e^{\frac{(\gamma|\gamma)}{2k}\delta}$ depend γ mod kM

$$F = \sum_{\gamma \in M^*} a_\gamma e^{\gamma + k\Lambda_0} = \sum_{\gamma \in M^* (\mathrm{mod}\ kM)} c_\gamma \sum_{\alpha \in \gamma + kM} e^{\alpha + k\Lambda_0 - \frac{(\alpha|\alpha)}{2k}\delta}$$

$\overline{\gamma + k\Lambda_0} = \gamma.$

$$\Rightarrow c_\gamma e^{-\frac{(\gamma|\gamma)}{2k}\delta} = \sum_{\gamma \in M^* (\mathrm{mod}\ kM)} c_\gamma(\tau)\, \Theta_{\gamma + k\Lambda_0}$$

$\lambda \in \mathfrak{h}^*, \langle \lambda, k \rangle = k$ $\quad \Theta_\lambda = \sum_{\beta \in \bar{\lambda} + kM} e^{k\Lambda_0 + \beta - \frac{|\beta|^2}{2k}\delta}$

$M = \sum_{i=1}^{l} \mathbb{Z}\alpha_i \subset \mathfrak{h}$

$M^* = \{ \lambda \in \mathfrak{h} \mid (\lambda|\alpha) \in \mathbb{Z} \text{ for } \alpha \in M \} \subset \mathfrak{h}$

$P_k = \{ \lambda \in \mathfrak{h} \mid \langle \lambda | \delta \rangle = k \text{ and } \bar{\lambda} \in M^* \}$

$\Downarrow$ $P_k = M^* + k\Lambda_0 + \mathbb{C}\delta.$

$$F = \sum_{\gamma \in P_k \bmod kM + \mathbb{C}\delta} c_\gamma(\tau)\, \Theta_\lambda \qquad (13.3.1)$$

Furthermore, fixed a positive real number a, then $\alpha \in k^{-1}M^*$

We have : $\Theta_\lambda(2\pi i\, \alpha + a\Lambda_0) = e^{2\pi i(\bar{\lambda}|\alpha)} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} e^{-\frac{1}{2}k(\gamma|\gamma)a}$

$\theta_\lambda \to$ $M^*/kM \to$ 特征.

$\theta_\lambda(2\pi i \alpha + a\wedge_0) = e^{k(\wedge_0 | 2\pi i \alpha + a\wedge_0)} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} e^{-\frac{1}{2}k(\gamma|\gamma)(\delta|2\pi i\alpha + a\wedge_0)} \, e^{k(\gamma|2\pi i\alpha + a\wedge_0)}$

$= e^{2\pi i(\bar{\lambda}|\alpha)} \sum_{\alpha \in M + k^{-1}\bar{\lambda}} \cdots$)

θ_λ 一维表示. 一特征标

F independent of u and for a fixed $\tau \in \mathcal{H} = \{\tau \in \mathbb{C}, \operatorname{Im}\tau > 0\}$

F 为 z 上 l 阶 某个函数. $|k^{-1}M^*/M| < \infty$

Since $(\cdot|\cdot)$ is nondegenerate: the characters of the group $k^{-1}M^*/M$ are linearly independent. We deduce:

$\{\theta_\lambda(\overset{\tau}{a}, z, 0) \mid \lambda \in P_k \bmod kM + \mathbb{C}\delta\}$ is a linearly independent set over $\mathbb{C}$, where θ_λ are ~~vieo~~ viewed as functions in z

$\chi \to$ $\mathfrak{h}$上 特征标

Finally, $F \in \widehat{Th}_k$, $D(F) = 0$

$$0 = D(F) = \frac{ik}{\pi} \sum_{\lambda \in |\cdots} \frac{d c_\gamma}{d\tau} \theta_\lambda$$

$\Rightarrow c_\gamma$ is constant in $\mathbb{C}$

$$D(f(\tau)\theta_\lambda) = D(\theta_\lambda) f(\tau) + D(f(\tau))\theta_\lambda + \frac{1}{2\pi}(f(\tau))\partial_u(\theta_\lambda)$$
$$= \frac{ki}{\pi} f'(\tau)\theta_\lambda$$

$\{\theta_\lambda \cdots \cdots Th_k$ for $k > 0\}$

Θ_λ $\widehat{\mathrm{Th}_p}$ $\longrightarrow$ $\widehat{\mathrm{Th}_0}$ — basis

Θ_λ

$F(u + a\delta) = F(u)$

$F(n\mathbb{Z} \cdot u) = F(u)$

Example: $M = \mathbb{Z}\alpha$ be a 1-dimensional lattice $\rightarrow (\alpha|\alpha) = 2$

$M^* = \frac{1}{2}M$, Thm $\longrightarrow$ basis

$|M^*/M| = 2m-1$

$\{\frac{\alpha}{\sqrt{2}}\}$ is the orthonormal.

v

$u = 2\pi i(z + u\delta + \tau/u_0)$

recall (13.2.5) $\Theta_\lambda(\tau, z, u) = e^{2\pi k i u} \sum_{\gamma \in M + k^{-1}\bar{\lambda}} e^{\pi i k \tau(\gamma|\gamma) + 2\pi i k(\gamma|z)}$

$\frac{\alpha n}{2} \in M^*$ $\lambda + t\delta$

$k = \langle \lambda, \delta \rangle = (\lambda|\delta) = m$

$\forall \lambda = \frac{\alpha}{2}n + m\Lambda_0$

$\in M^*$

$\Theta_{n,m} = \Theta_{\frac{n\alpha}{2} + m\Lambda_0} = e^{2\pi i m u} \sum_{\gamma \in \mathbb{Z}\alpha + \frac{n\alpha}{m2}} e^{\pi i m \tau(\gamma|\gamma) + 2\pi i m(\gamma|z)}$

$n = \{0, \ldots, 2m-1\}$

$M = \mathbb{Z}\alpha$

$\frac{\alpha n}{2m} \in \mathbb{Z}\alpha$

$\frac{n}{2m} \notin \mathbb{Z}$

$= e^{2\pi i m u} \sum_{k \in \mathbb{Z} + \frac{n}{2m}} e^{\pi i m(\tau k^2 + kz)}$

$\gamma = \frac{\alpha}{\sqrt{2}} k$

$n \in \mathbb{Z}$ mod $2m\mathbb{Z}$

$n = \overline{0,}\ 2m-1.$

Sec 13.4 $\quad SL_2(\mathbb{Z}) \to$ 完全模群

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \quad ad-bc=1 \right\}$$

$$\mathcal{H} = \{ \tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0 \}$$

$SL_2(\mathbb{R})$ operates on $\mathcal{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d} \in \mathcal{H}$$

$\forall n \in \mathbb{Z}_+ \setminus \{0\}$

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{array}{l} a \equiv d \equiv 1 \mod n \\ b \equiv c \equiv 0 \mod n \end{array} \right\}$$

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod n \right\}$$

$$\Gamma_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; ac \text{ and } bd \text{ are even} \right\}$$

$$\left| SL_2(\mathbb{Z}) \big/ \Gamma(n),\ \Gamma_0(n),\ \Gamma_\theta \right| < \infty$$

$$SL_2(\mathbb{Z}) \Leftarrow \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\Gamma \ni \gamma, \quad \gamma = T^{n(1)} S \; T^{n(2)} S \cdots T^{n(k-1)} S \; T^{n(k)}$$

- $S^2 = -I = T^0 S T^0 S \quad (k=2)$
- $c = 0 \quad \gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = T^b$

  $d > 0$

- Suppose $c \geq 0$, by induction on c
- $c=0$ ✓, $k > c \geq 0$ ✓
- when $c=k$, $d = qk + r$ $\quad -k < r \leq 0$

$$\tau T^{-q} S^{-1} = \begin{pmatrix} a & b \\ k & d \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} -b+qa & a \\ -r & k \end{pmatrix}}$$

$-r < k$

$$\tau = T^q \,\underline{\triangle}\, S$$

Similarly: $\langle S, T^2 \rangle = \Gamma_\theta$

$\langle T, ({}^tT)^r = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, -I \rangle = \Gamma_0(r)$ for $r = 2$ or 3

- Recall that the metaplectic group $Mp_2(\mathbb{R})$ is a double cover of $SL_2(\mathbb{R})$:

$$Mp_2(\mathbb{R}) = \{ (A, j) \mid A \in SL_2(\mathbb{R}),\ \underline{j^2 = c\tau + d} \}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- multi: $(A, j) \cdot (A_1, j_1) = (AA_1, \underline{j_{AA_1}(\tau)} = j_A(A_1 \cdot \tau) j_{A_1}(\tau))$

$$\pm \sqrt{c\left(\frac{a_1\tau + b_1}{c_1\tau + d_1}\right) + d} \sqrt{c_1\tau + d_1}$$

$$= \pm \sqrt{c(a_1\tau + b_1) + d(c_1\tau + d_1)}$$

$$= j_{AA_1}(\tau)$$

- 单位元: $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ j = 1 \right)$

…

$Mp_2(\mathbb{Z})$ $\quad Mp_2^\theta(\mathbb{Z}) = \{ (A, j) \in Mp_2(\mathbb{R}) \mid A \in \Gamma_\theta \}$

$Mp_2(\mathbb{R}) \times Y \longrightarrow Y$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z, u) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau+d}, u - \frac{c(z|z)}{2(c\tau+d)} \right)$$

$$\begin{cases} A_1[A(\tau, z, u)] = (A_1 A)(\tau, z, u) \\ I_2(\tau, z, u) = (\tau, z, u) \end{cases}$$

where $z \in \mathfrak{h}^0$, $\tau \in H$, $u \in \mathbb{C}$

claim: $Mp_2(\mathbb{R})$ normalizes $N = \mathfrak{h}^0_{\mathbb{R}} \times \mathfrak{h}^0_{\mathbb{R}} \times i\mathbb{R}$.

(13.4.3) $\quad \left((A, j)(\alpha, \beta, u)(A, j)^{-1} \right) = (a\alpha + b\beta, c\alpha + d\beta, u) \in N$

$(A, j) \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as holomorphic automorphism of Y

fact: $\begin{cases} A(\alpha, 0, 0)A^{-1}(\tau, z, u) = (a\alpha, c\alpha, 0)(\tau, z, u) \\ A(0, \beta, 0)A^{-1}(\tau, z, u) = (b\beta, d\beta, 0)(\tau, z, u) \\ A(0, 0, u)A^{-1} = (0, 0, u) \end{cases}$

recall $(\alpha, \beta, u)(\alpha', \beta', u') = (\alpha + \alpha', \beta + \beta', u + u' + \pi i((\alpha|\beta') - (\alpha'|\beta)))$

$$A(\alpha, \beta, u)A^{-1} = A\left((\alpha, 0, 0)(0, \beta, 0)(0, 0, u - 2\pi i(\alpha|\beta)\right)A^{-1}$$

$$= A(\alpha, 0, 0)A^{-1} \quad A(0, \beta, 0)A^{-1} \quad A(0, 0, u - 2\pi i(\alpha|\beta))A^{-1}$$

$$= (a\alpha, c\alpha, 0)(b\beta, d\beta, 0)(0, 0, u - \pi i(\alpha|\beta))$$

$= (a\alpha + b\beta,\ c\alpha + d\beta,\ u)$ $\quad ad - bc = 1$

Hence, we have an action of the group $G := M_{\rho_2}(\mathbb{R}) \ltimes N$ on Y.

$$M_{\rho_2}(\mathbb{Z}) \longrightarrow N_{\mathbb{Z}} = \{ (\alpha.\beta.u \in N \mid \alpha.\beta \in M,\ u + \pi i (\alpha|\beta) \in 2\pi i \mathbb{Z} \}$$

prop: The normalizer of $N_{\mathbb{Z}}$ in the subgroup $M_{\rho_2}(\mathbb{R})$ of G is $M_{\rho_2}(\mathbb{Z})$ if the lattice M is even ($\forall \gamma \in M$, $(\gamma|\gamma)$ are even), $M^{\theta}_{\rho_2}(\mathbb{Z})$ if M is odd

pf: $A \in SL_2(\mathbb{Z})$ $\quad A N_{\mathbb{Z}} A^{-1} \in N_{\mathbb{Z}}$ $\quad \overset{?}{\in} N_{\mathbb{Z}}$

$$A(\alpha.\beta.u)A^{-1} = (a\alpha + b\beta,\ c\alpha + d\beta,\ u)$$

$$u + \pi i (a\alpha + b\beta \mid c\alpha + d\beta) \overset{?}{\in} 2\pi i \mathbb{Z}$$

$$\|$$

$$= u + \pi i \big(ac(\alpha|\alpha) + bd(\beta|\beta) + (\alpha|\beta) + 2bc(\alpha|\beta)\big)$$

$\in 2\pi i \mathbb{Z}$

when $\pi i \big(ac(\alpha|\alpha) + bd(\beta|\beta)\big) \in 2\pi i \mathbb{Z}$ $\quad (\alpha|\alpha), (\beta|\beta) \in 2\mathbb{Z}$

for M is even $(\alpha|\alpha), (\beta|\beta) \in 2\mathbb{Z}$

$$N_{\mathbb{Z}} = \{ (\alpha.\beta.u) \quad \alpha.\beta \in M = \sum_{i=1}^{l} \mathbb{Z}\alpha_i,\ u + \pi i (\alpha|\beta) \in 2\pi i \mathbb{Z} \}$$

$$A(\alpha.\beta.u)A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\alpha.\beta.u) \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = (a\alpha + b\beta,\ c\alpha + d\beta,\ u)$$

$$\in N_{\mathbb{Z}} \iff a\alpha + b\beta \in M,\ c\alpha + d\beta \in M$$

and $\underline{u + 2\pi i(a\alpha + b\beta} \mid c\alpha + d\beta) \in 2\pi i\mathbb{Z}$

i.e $a, b, c, d \in \mathbb{Z}$ and M is even or

$a, b, c, d \in \mathbb{Z}$, and ac, bd are even.

i.e $A \in \Gamma_\theta$ #